A Systems Theory Approach to the Synthesis of Minimum Noise Non-Reciprocal and Phase-Insensitive Quantum Amplifiers

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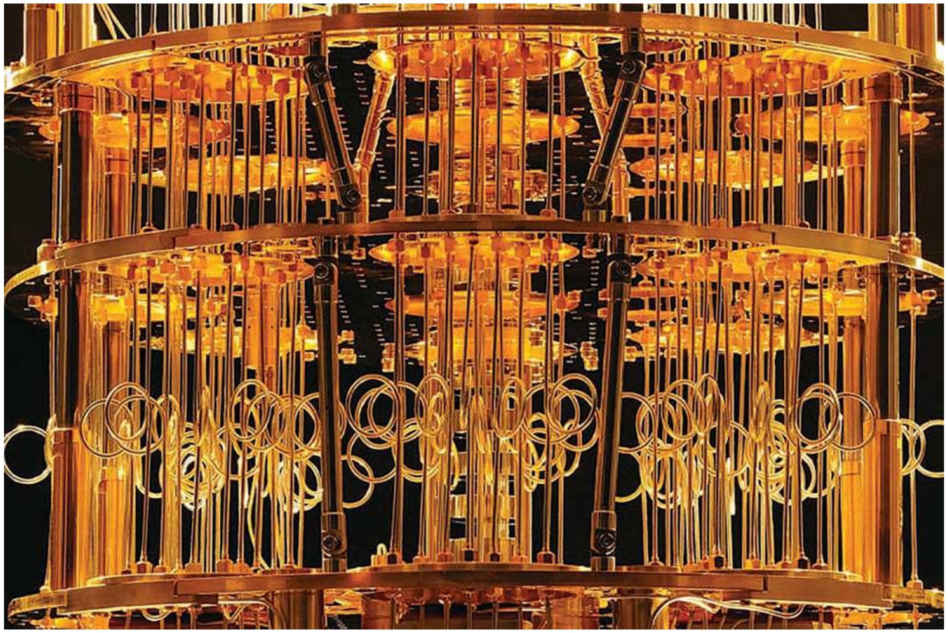
Introduction

- Developments in quantum technology and quantum information provide an important motivation for research in the area of quantum control systems.
- One of the most significant areas of long term opportunity in quantum technology is that of quantum computing.



Superconducting quantum computing experiment

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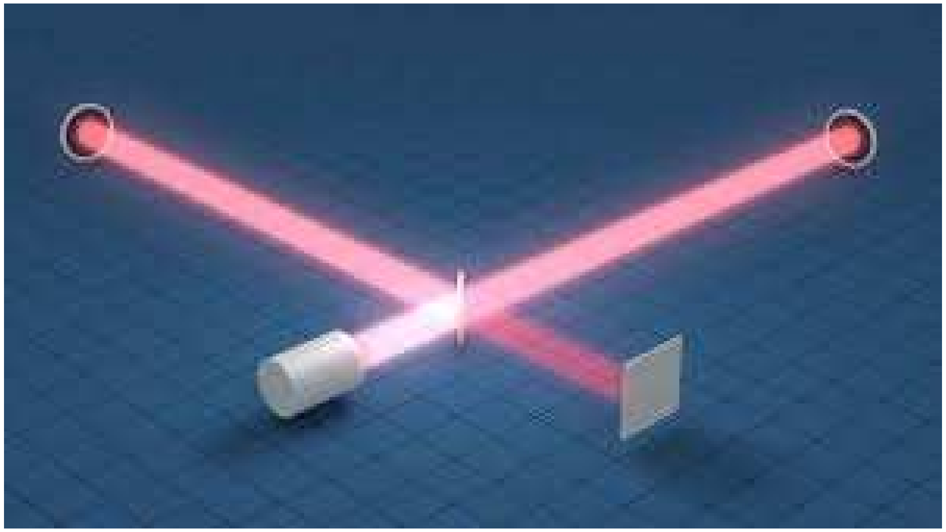
- Companies such as Google, IBM, and Microsoft have made significant investments in quantum computing to develop small scale quantum computers using microwave frequency technologies involving arrays of superconducting Josephson junctions operating at millikelvin temperatures.
- Recently Google claimed to achieve “Quantum Supremacy” with their technology, indicating that their quantum computer could solve a problem which was impossible to solve with a classical computer, although other companies have disputed this claim.
- Quantum amplifiers play a critical role in such quantum superconducting technologies in that they are required to read out qubit states and transfer the information to the classical world at room temperature.

- Another important area of quantum technology is quantum sensing.
- One of the most significant achievements in the area of quantum sensing is the detection of gravitational waves.
- In this case, a quantum optics technology has been used and optical quantum amplification is required to extract the extremely feint gravity wave signals.



Schematic of the LIGO gravity wave detection experiment

- It is well known in the physics literature that the laws of quantum mechanics place fundamental limits on quantum amplifiers in terms of the amount of noise which is added by the amplifier to a signal in order to achieve a given level of amplification.
- We will re-derive those limits using quantum linear systems theory and investigate problems of designing optimal quantum amplifiers.
- We will first consider phase-insensitive quantum amplifiers which provide amplification while maintaining the coherent structure of the quantum signal.
- We then consider non-reciprocal phase-insensitive quantum amplifiers which have the additional property that they eliminate quantum back-action from the amplifier to the quantum system being measured.

Quantum Linear Systems

- Quantum linear systems are a class quantum system models in the **Heisenberg Picture** of quantum mechanics which describes the time evolution of operators representing system variables such as position and momentum.
- This is as opposed to the Schrödinger picture which describes quantum systems in terms of the time evolution of the quantum state.

Werner Heisenberg

- We formulate a class of linear quantum system models described by quantum stochastic differential equations (QSDEs) derived from the quantum harmonic oscillator.
- We begin by considering a collection of n independent quantum harmonic oscillators which are defined on a Hilbert space $\mathscr{H}$.
- Corresponding to this is a vector of *annihilation operators* a:

$$a = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}.$$

- Each annihilation operator a_i is an unbounded linear operator on $\mathscr{H}$.

- The adjoint of the operator a_i is denoted a_i^* and is referred to as a *creation operator*. We use $a^\#$ to denote the vector of a_i^*s:

$$a^\# = \begin{bmatrix} a_1^* \\ a_2^* \\ \vdots \\ a_n^* \end{bmatrix}.$$

- Physically, these operators correspond to the annihilation and creation of a photon respectively.

- Also, we use the notation $a^T = \begin{bmatrix} a_1 & a_2 & \ldots & a_n \end{bmatrix}$, and $a^\dagger = \left(a^\#\right)^T = \begin{bmatrix} a_1^* & a_2^* & \ldots & a_n^* \end{bmatrix}$.

- Matrices of the form $\begin{bmatrix} R_1 & R_2 \\ R_2^\# & R_1^\# \end{bmatrix}$ are denoted by $\Delta(R_1, R_2)$.

- Also, $J := \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$.

- We consider a class of linear quantum systems described by the QSDEs:

$$\begin{bmatrix} da(t) \\ da(t)^{\#} \end{bmatrix} = A \begin{bmatrix} a(t) \\ a(t)^{\#} \end{bmatrix} dt + B \begin{bmatrix} du(t) \\ du(t)^{\#} \end{bmatrix};$$
$$\begin{bmatrix} dy(t) \\ dy(t)^{\#} \end{bmatrix} = C \begin{bmatrix} a(t) \\ a(t)^{\#} \end{bmatrix} dt + D \begin{bmatrix} du(t) \\ du(t)^{\#} \end{bmatrix},$$

where

$$A = \Delta(A_1, A_2), \quad B = \Delta(B_1, B_2),$$
$$C = \Delta(C_1, C_2), \quad D = \Delta(D_1, D_2).$$

- Here, $a(t) = [a_1(t) \cdots a_n(t)]^T$ is a vector of annihilation operators corresponding to each quantum harmonic oscillator in the system, the vector u represents the input signals and the vector y represents the output signals.

Definition 1 *A complex linear quantum system of the above form is said to be* **physically realizable** *if there exists a complex Hamiltonian matrix* $M = M^\dagger$*, a coupling matrix* N *and a unitary scattering matrix* S *such that* M *and* N *are of the form* $M = \Delta(M_1, M_2)$*,* $N = \Delta(N_1, N_2)$ *and*

$$\begin{aligned} A &= -\imath JM - \frac{1}{2}JN^\dagger JN; \\ B &= -JN^\dagger J; \\ C &= N; \\ D &= \begin{bmatrix} S & 0 \\ 0 & S^\# \end{bmatrix}. \end{aligned}$$

- The square complex transfer function matrix corresponding to the above system is given by

$$G(s) = C(sI - A)^{-1}B + D.$$

Definition 2 *A complex transfer function matrix $G(s)$ is said to be* **physically realizable** *if it is the transfer function of a physically realizable linear quantum system.*

Theorem 1 *A square complex transfer function matrix $G(s)$ is physically realizable if and only if*

$$G^{\sim}(s)JG(s) = J$$

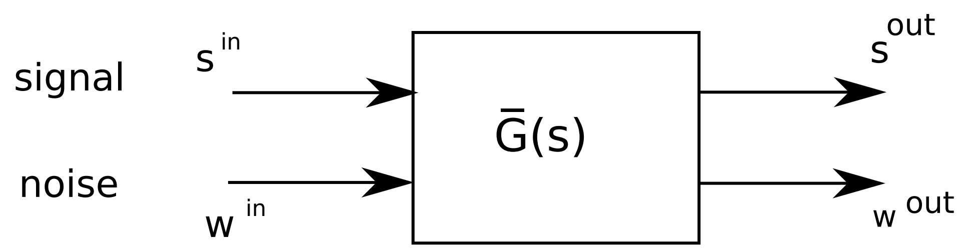
for all $s \in \mathbb{C}$ and the matrix $G(\infty)$ is of the form $G(\infty) = \begin{bmatrix} S & 0 \\ 0 & S^{\#} \end{bmatrix}$ where $S^{\dagger}S = SS^{\dagger} = I$. Here, $G^{\sim}(s) = G(-s^{*})^{\dagger}$.

- A physically realizable transfer function matrix corresponds to a linear quantum system which satisfies the laws of quantum mechanics and can be implemented using physical components such as arising in quantum optics or quantum superconducting circuits.

Phase-Insensitive Quantum Amplifier

- In quantum optics, all signals have two components or quadratures. In amplifying a quantum signal, it is usually a requirement that both quadratures of the signal be amplified equally.
- An amplifier having this property is called a phase-insensitive quantum amplifier.
- We present a systems theory approach to the proof of a result bounding the required level of added quantum noise in a phase-insensitive quantum amplifier.
- We also present a synthesis procedure for constructing a quantum optical phase-insensitive quantum amplifier which adds the minimum level of quantum noise and achieves a required gain and bandwidth.
- This synthesis procedure is based on a singularly perturbed quantum system and leads to an amplifier involving two squeezers and two beamsplitters in the optical case.

- A phase-insensitive quantum amplifier is a two-input two-output physically realizable quantum linear system with transfer function $\bar{G}(s)$ as illustrated below:



- In this diagram, the first input channel and the first output channel are the signal input and output channels respectively.
- Also, the second input channel and the second output channel are noise input and output channels.

- The noise output channel is not used in the operation of the amplifier but is included for consistency with the physical realizability theory for quantum linear systems.
- As with any quantum linear system, each input and output channel consists of two quadratures.
- Hence, the transfer function matrix $\bar{G}(s)$ is a four-by-four transfer function matrix.
- In order to define a phase-insensitive quantum amplifier, a physically realizable transfer function matrix $\bar{G}(s)$ should satisfy certain gain and phase-insensitivity properties over a specified frequency range.

- We write the transfer function $\bar{G}(s)$ in "doubled-up" form, specifying both quadratures of each input and output channels as follows:

$$\begin{bmatrix} s^{out} \\ w^{out} \\ s^{out*} \\ w^{out*} \end{bmatrix} = \bar{G} \begin{bmatrix} s^{in} \\ w^{in} \\ s^{in*} \\ w^{in*} \end{bmatrix} = \begin{bmatrix} G & H \\ H^{\#} & G^{\#} \end{bmatrix} \begin{bmatrix} s^{in} \\ w^{in} \\ s^{in*} \\ w^{in*} \end{bmatrix}.$$

- Furthermore, we write

$$G = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}, \quad H = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}.$$

Definition 3 *A physically realizable transfer function matrix $\bar{G}(s)$ of the form above is said to be* **phase-insensitive** *at frequency ω if*

$$h_{11}(j\omega) = 0.$$

Definition 4 *A physically realizable transfer function matrix $\bar{G}(s)$ of the form above is said to have* **complex gain** *g at frequency ω if*

$$g_{11}(j\omega) = g.$$

- We will be mostly concerned with the phase-insensitive property at DC and hence, we will usually drop the frequency specification.
- Also, we will be concerned with the *noise* squared amplitude

$$g_{12}(j\omega)^* g_{12}(j\omega) + h_{12}(j\omega)^* h_{12}(j\omega)$$

at a given frequency ω (usually DC).

Main Results

Theorem 2 *At any frequency ω, given a desired phase-insensitive quantum amplifier gain at that frequency g, then the minimum possible value of the noise squared amplitude is*

$$\min\left[g_{12}^* g_{12} + h_{12}^* h_{12}\right] = g^* g - 1.$$

Here the minimum is taken over all transfer function matrices $\bar{G}(s)$ satisfying the physical realizability condition, the phase-insensitivity condition and with the given amplifier gain g. Furthermore, this minimum is achieved by a transfer function matrix defined by

$$\begin{aligned} g_{12} &= 0; g_{21} = \sqrt{\frac{g^* g - 1}{g^* g}}; g_{22} = \sqrt{1 + g^* g}; \\ h_{11} &= 0; h_{12} = \frac{1}{g^*}\sqrt{g^* g\left(g^* g - 1\right)}; \\ h_{21} &= \sqrt{\frac{g^* g - 1}{g^* g}}; h_{22} = 1. \end{aligned}$$

However, this minimum is not unique.

- To apply this theorem, we use the following result which is known as the Shale decomposition.

Lemma 1 *Consider a* 4×4 *complex matrix* $\bar{G}$ *of the form above satisfying the physical realizability condition. Then there exists a real diagonal matrix* $R = \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix}$ *and* 2×2 *unitary matrices* S_1 *and* S_2 *such that*

$$\bar{G} = \begin{bmatrix} S_1 & 0 \\ 0 & S_1^{\#} \end{bmatrix} \begin{bmatrix} -\cosh(R) & -\sinh(R) \\ -\sinh(R) & -\cosh(R) \end{bmatrix} \begin{bmatrix} S_2 & 0 \\ 0 & S_2^{\#} \end{bmatrix}.$$

- This lemma shows that the problem of physically realizing the two channel DC gain transfer function matrix $\bar{G}$ can be reduced to the problem of physically realizing each of the single channel transfer function matrices $\bar{G}_1 = \begin{bmatrix} -\cosh(r_1) & -\sinh(r_1) \\ -\sinh(r_1) & -\cosh(r_1) \end{bmatrix}$, and $\bar{G}_2 = \begin{bmatrix} -\cosh(r_2) & -\sinh(r_2) \\ -\sinh(r_2) & -\cosh(r_2) \end{bmatrix}$.

- Then, the unitary transfer matrices $\begin{bmatrix} S_1 & 0 \\ 0 & S_1^{\#} \end{bmatrix}$, and $\begin{bmatrix} S_2 & 0 \\ 0 & S_2^{\#} \end{bmatrix}$ can be physically implemented using beamsplitters. Indeed, since S_1 and S_2 are both 2×2 matrices, it follows that each of these can be implemented by a single beamsplitter.

- For example, we can write the input-output relations of a beamsplitter in the form

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathscr{R} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

  where $\mathscr{R}$ is a unitary matrix of the form

$$\mathscr{R} = \begin{bmatrix} e^{j\phi_1} \sin(\theta) & e^{j(\phi_1+\phi_3)} \cos(\theta) \\ e^{j\phi_2} \cos(\theta) & -e^{j(\phi_2+\phi_3)} \sin(\theta) \end{bmatrix}$$

  and ϕ_1, ϕ_2, ϕ_3 and θ are parameters of the beamsplitter.

- Furthermore, it is straightforward to verify that any 2×2 unitary matrix S can be represented as a matrix of this form.

- To realize a single channel DC transfer function matrix

$$\bar{G}_r = \begin{bmatrix} -\cosh(r) & -\sinh(r) \\ -\sinh(r) & -\cosh(r) \end{bmatrix},$$

  we consider a single channel dynamic squeezer.

- An optical cavity consists of a number of mirrors, one of which is partially reflective. If we include a nonlinear optical element inside such a cavity, an optical squeezer can be obtained. By using suitable linearizations and approximations, such an optical squeezer can be described by a quantum stochastic differential equation as follows:

$$\begin{aligned} da &= -\frac{\kappa}{2} a dt - \chi a^* dt - \sqrt{\kappa} du; \\ dy &= \sqrt{\kappa} a dt + du, \end{aligned}$$

  where $\kappa > 0$, χ is a complex number associated with the strength of the nonlinear effect and a is a single annihilation operator associated with the cavity mode.

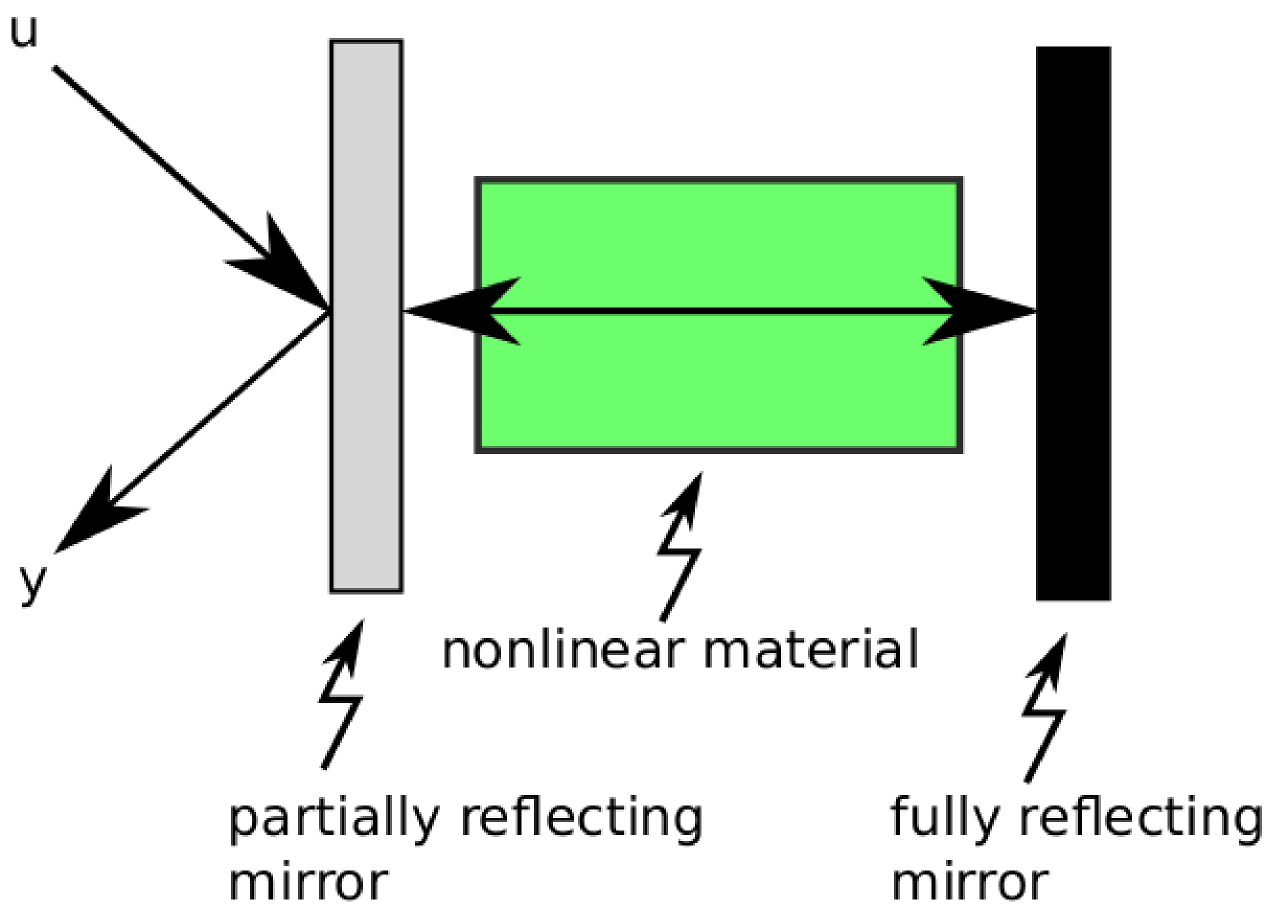
- This leads to a linear quantum system as follows:

$$\begin{bmatrix} da(t) \\ da(t)^* \end{bmatrix} = \begin{bmatrix} -\frac{\kappa}{2} & -\chi \\ -\chi^* & -\frac{\kappa}{2} \end{bmatrix} \begin{bmatrix} a(t) \\ a(t)^* \end{bmatrix} dt - \sqrt{\kappa} \begin{bmatrix} du \\ du^* \end{bmatrix};$$

$$\begin{bmatrix} dy \\ dy^* \end{bmatrix} = \sqrt{\kappa} \begin{bmatrix} a(t) \\ a(t)^* \end{bmatrix} dt + \begin{bmatrix} du \\ du^* \end{bmatrix}.$$

- Note that it is straightforward to verify that this system is stable if and only if $\kappa^2 > 4\chi\chi^*$.

- A diagram of a dynamic optical squeezer is shown below:



- Now, we choose the parameters κ and χ to be of the form $\kappa = \epsilon\bar{\kappa}$ and $\chi = \epsilon\bar{\chi}$, where $\bar{\kappa} > 0$, $\bar{\chi}$ is chosen to be real and $\epsilon > 0$ is a parameter which will determine the amplifier bandwidth.

- Introducing the change of variables $\begin{bmatrix} \tilde{a}(t) \\ \tilde{a}(t)^* \end{bmatrix} = \epsilon^{-\frac{1}{2}} \begin{bmatrix} a(t) \\ a(t)^* \end{bmatrix}$, the above QSDEs reduce to

$$\begin{bmatrix} d\tilde{a}(t) \\ d\tilde{a}(t)^* \end{bmatrix} = \frac{1}{\epsilon}\begin{bmatrix} -\frac{\bar{\kappa}}{2} & -\bar{\chi} \\ -\bar{\chi} & -\frac{\bar{\kappa}}{2} \end{bmatrix}\begin{bmatrix} \tilde{a}(t) \\ \tilde{a}(t)^* \end{bmatrix} dt - \frac{\sqrt{\bar{\kappa}}}{\epsilon}\begin{bmatrix} du \\ du^* \end{bmatrix};$$

$$\begin{bmatrix} dy \\ dy^* \end{bmatrix} = \sqrt{\bar{\kappa}}\begin{bmatrix} \tilde{a}(t) \\ \tilde{a}(t)^* \end{bmatrix} dt + \begin{bmatrix} du \\ du^* \end{bmatrix}.$$

- The transfer function matrix of this system at DC is given by

$$G(0) = I + \begin{bmatrix} -\frac{2\bar{\kappa}^2}{\bar{\kappa}^2-4\bar{\chi}^2} & \frac{4\bar{\kappa}\bar{\chi}}{\bar{\kappa}^2-4\bar{\chi}^2} \\ \frac{4\bar{\kappa}\bar{\chi}}{\bar{\kappa}^2-4\bar{\chi}^2} & -\frac{2\bar{\kappa}^2}{\bar{\kappa}^2-4\bar{\chi}^2} \end{bmatrix} = \begin{bmatrix} -\frac{1+\alpha^2}{1-\alpha^2} & \frac{2\alpha}{1-\alpha^2} \\ \frac{2\alpha}{1-\alpha^2} & -\frac{1+\alpha^2}{1-\alpha^2} \end{bmatrix}$$

  where $\alpha = \frac{2\bar{\chi}}{\bar{\kappa}} = \frac{2\chi}{\kappa}$.

- After some manipulation, we obtain the following result.

Lemma 2 *Given any matrix G_r of the form above, there exists a physically realizable quantum system corresponding to a stable single channel dynamic squeezer such that its transfer function matrix $G(s)$ satisfies*

$$G(0) = G_r.$$

Here, the ratio $\alpha = \frac{2\bar{\chi}}{\bar{\kappa}}$ satisfies $\alpha^2 < 1$ and the parameter $\epsilon > 0$ can be chosen to achieve any desired bandwidth.

- Using the above theorems and lemmas, we obtain the following theorem which is one of our main results.

Theorem 3

- *Given any desired quantum phase-insensitive amplifier DC gain g, there exists a corresponding physically realizable linear quantum system which achieves this DC gain and introduces the minimal amount of DC quantum noise.*
- *Furthermore, this transfer function matrix satisfies the DC phase-insensitivity condition.*
- *In addition, the parameters in this linear quantum system can be chosen to achieve a specified bandwidth over which the above conditions will hold approximately.*
- *Finally, this system can be constructed from two beamsplitters and two stable dynamic squeezers.*

Illustrative Example

- We now apply the method of this paper to synthesise a phase-insensitive quantum amplifier with a DC gain of $g = 2$ (6dB), a bandwidth of 2×10^6 radians/s and with the minimum added noise.
- Indeed, with $g = 2$, our formulas give

$$G = \begin{bmatrix} 2 & 0 \\ \frac{\sqrt{3}}{2} & \sqrt{5} \end{bmatrix}; \quad H = \begin{bmatrix} 0 & \sqrt{3} \\ \frac{\sqrt{15}}{2} & 1 \end{bmatrix}.$$

- We then obtain

$$R = \begin{bmatrix} 1.6139 & 0 \\ 0 & -1.1327 \end{bmatrix}.$$

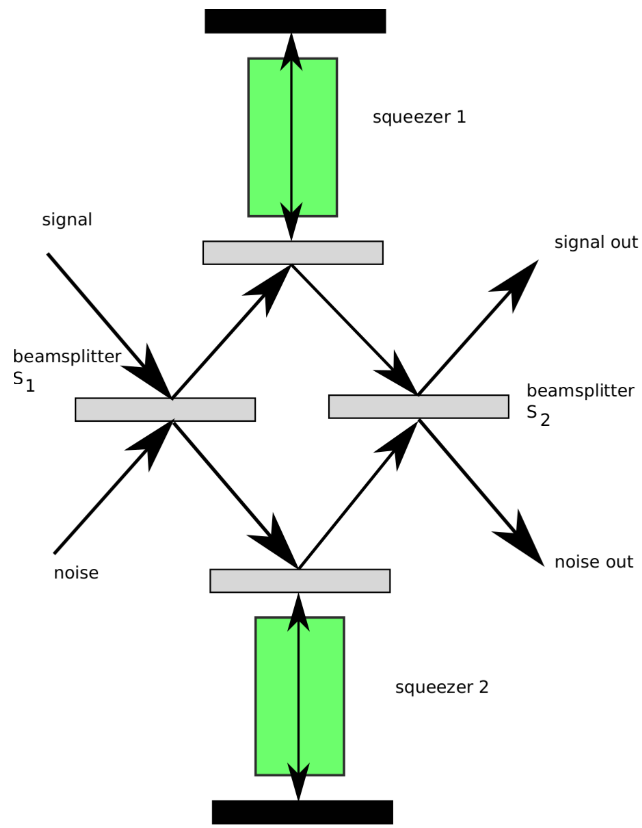
- Also, we have

$$S_1 = \begin{bmatrix} 0.5240 & 0.8517 \\ 0.8517 & -0.5240 \end{bmatrix}$$

and

$$S_2 = \begin{bmatrix} -0.6840 & -0.7295 \\ -0.7295 & 0.6840 \end{bmatrix}.$$

- These parameter values define the beamsplitters representing the matrices S_1 and S_2 respectively.
- Also, the matrix R defines the parameters $\alpha_1 = -0.6679$ and $\alpha_2 = 0.5127$.
- These parameters are then used to define the parameters for the two squeezers. First we choose the parameter $\epsilon = 2\pi 10^6$ radians/s to achieve the specified bandwidth.
- Then, we choose the parameters $\kappa_1 = 2\pi * 10^6$ radians/s, $\chi_1 = \frac{\alpha_1 \kappa_1}{2} = -2.0983 \times 10^6$ radians/s for the first squeezer, and the parameters $\kappa_2 = 2\pi * 10^6$ radians/s, $\chi_2 = \frac{\alpha_2 \kappa_2}{2} = 1.6106 \times 10^6$ radians/s for the second squeezer.
- The implementation of the phase-insensitive amplifier is as shown below.

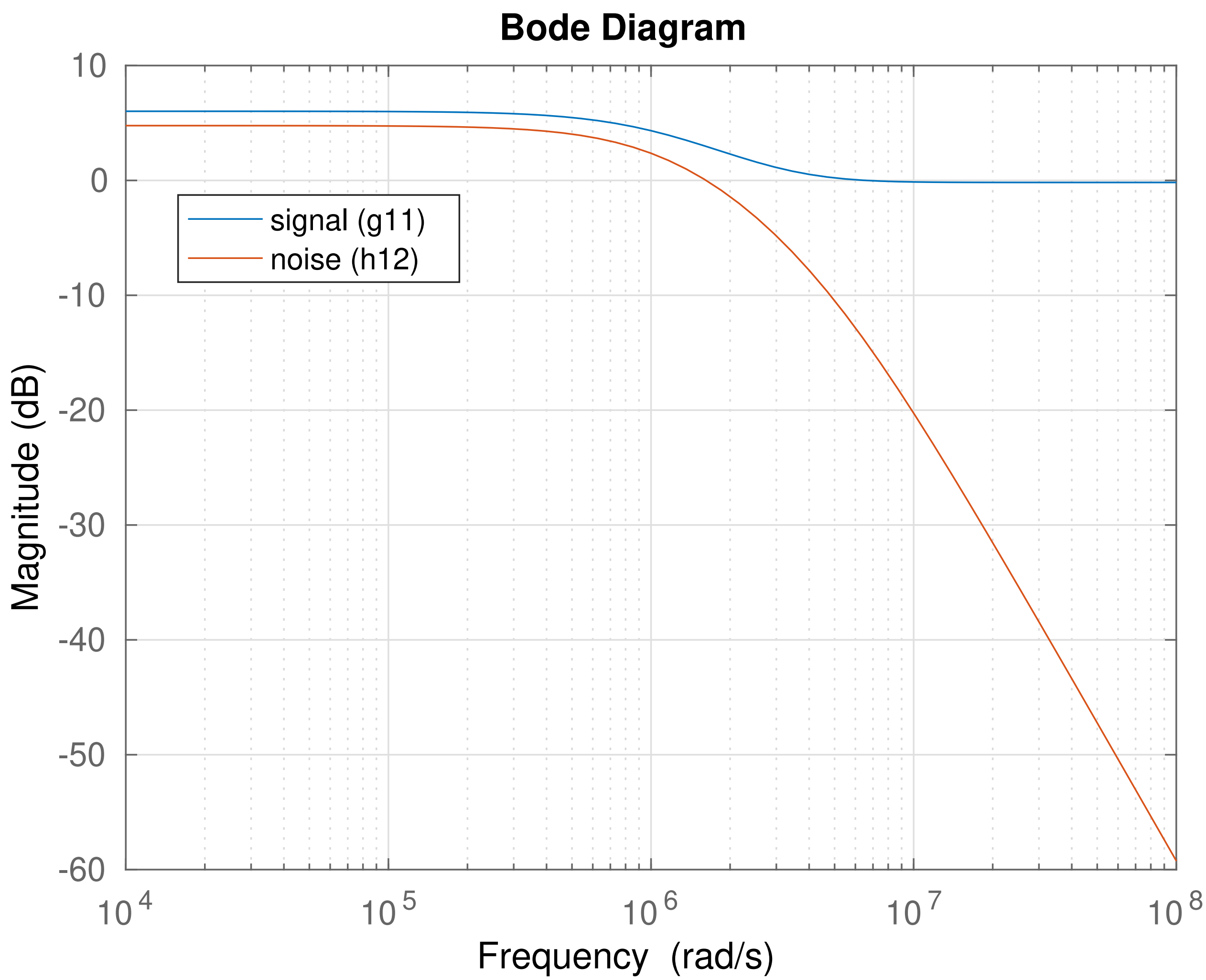


- Proposed realization of the phase-insensitive quantum amplifier.

- We now calculate the transfer function matrix of this proposed phase-insensitive quantum amplifier.
- Let $\tilde{G}_1(s)$ be the transfer function of the first squeezer and let $\tilde{G}_2(s)$ be the transfer function of the second squeezer.
- Then the transfer function matrix of the overall phase-insensitive quantum amplifier system is given by

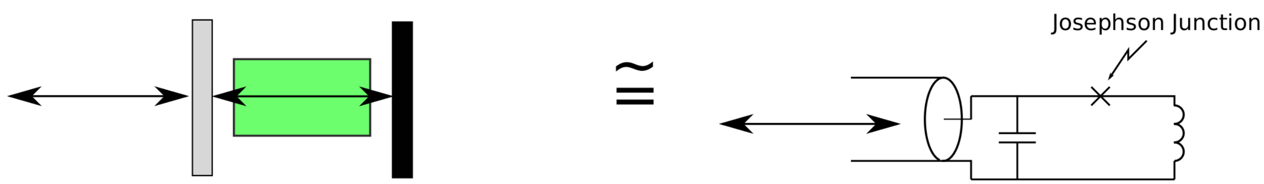
$$\bar{G}(s) = \begin{bmatrix} S_1 & 0 \\ 0 & S_1^{\#} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{G}_1(s) & 0 \\ 0 & \tilde{G}_2(s) \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} S_2 & 0 \\ 0 & S_2^{\#} \end{bmatrix}.$$

- We construct this transfer function matrix for this example and then plot the magnitude Bode plot of the $(1,1)$ block of $\bar{G}(s)$ as shown below.
- This is the transfer function from the signal input to the signal output $g(s)$.
- This plot also shows the magnitude Bode plot of the $(1,4)$ block of $\bar{G}(s)$.
- This is the transfer function $h_{12}(s)$ from the quadrature noise input to the signal output.
- This plot shows that at DC, the amplifier gives 6 dB of gain but there is a noise signal which is of a magnitude given by $\sqrt{g(0)^2 - 1}$.


Bode Diagram
10
0
-10
-20
-30
-40
-50
-60
Magnitude (dB)
signal (g11)
noise (h12)
10^4
10^5
10^6
10^7
10^8
Frequency (rad/s)

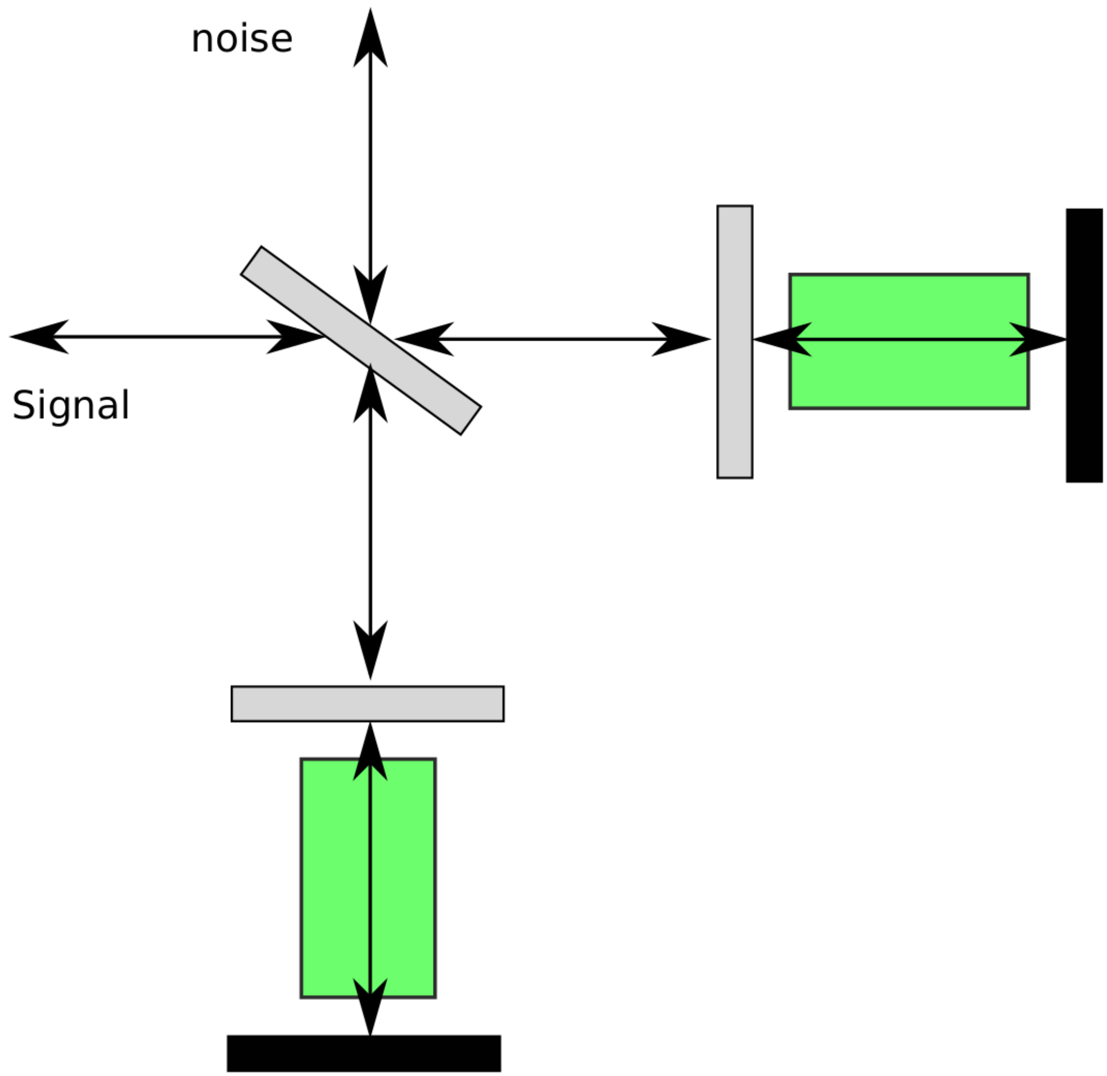
Realization of a Phase-Insensitive Amplifier using Microwave Circuits

- A superconducting microwave equivalent of an optical squeezer can be constructed using a Josephson junction as illustrated below.



- A significant limitation in this case is that the input and output of the squeezer are always coincident as forward and reflected waves in the transmission line.
- This would force the optical equivalent of the phase-insensitive amplifier to have the same beamsplitter used in both the forward and reflected directions.

- This is illustrated below in the optical case.



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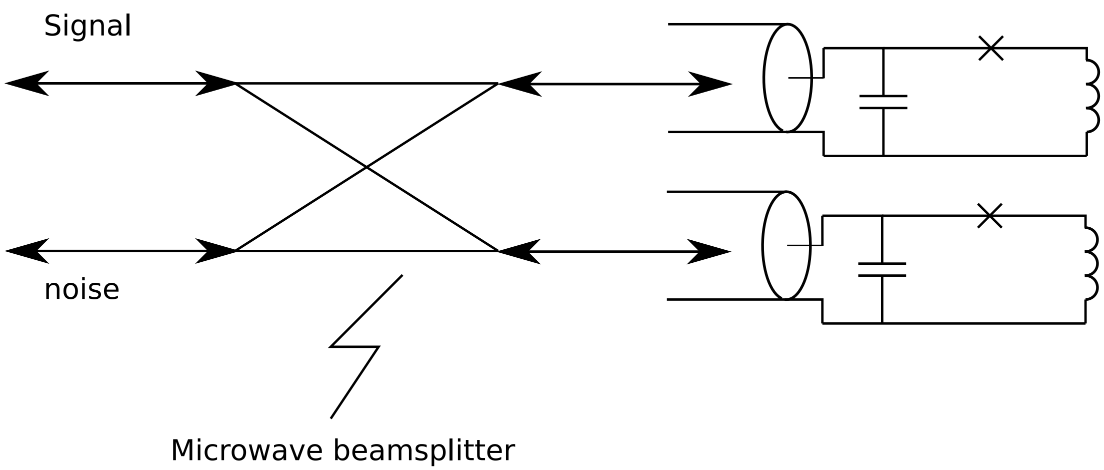
- In the microwave case, the corresponding amplifier structure is as follows.



- This structure puts an extra restriction on the set of possible amplifier transfer function matrices in that the orthogonal matrices in the Shale decomposition must be equal $S_1 = S_2$.

- In recent work, we have found that it is possible to find a phase-insensitive quantum amplifier transfer function matrix which satisfies this extra condition and still achieves the minimum possible noise level of

$$\min\left[g_{12}^* g_{12} + h_{12}^* h_{12}\right] = g^* g - 1.$$

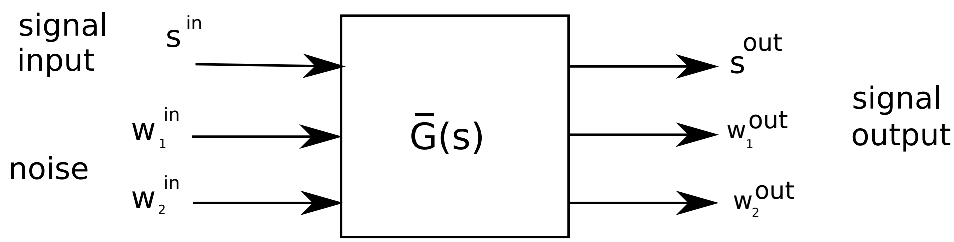
- This was done by using the following symmetric transfer function matrix $\bar{G}$:

$$\bar{G} = \begin{bmatrix} g & 0 & 0 & \sqrt{g^2-1} \\ 0 & g & \sqrt{g^2-1} & 0 \\ 0 & \sqrt{g^2-1} & g & 0 \\ \sqrt{g^2-1} & 0 & 0 & g \end{bmatrix}.$$

Non-Reciprocal Phase-Insensitive Quantum Amplifier

- It is often important that a phase insensitive amplifier has the non-reciprocal property.
- This means that when an optical signal is applied to the input port of the amplifier, none of that signal emerges from the same port.
- In the case of optical amplifiers which do not have the non-reciprocal property, the reflected signal may damage the device which was generating the optical signal.
- Similar issues arise in non-optical quantum technologies such as superconducting microwave circuits. These problems can be addressed by introducing an isolator or a circulator but such devices may be noisy and difficult to implement.
- Hence, the problem of constructing a non-reciprocal phase-insensitive quantum amplifiers has received growing attention.

- A non-reciprocal phase-insensitive quantum amplifier is a three-input three-output physically realizable quantum linear system with transfer function $\bar{G}(s)$ as illustrated below:



- In this diagram, the first input channel and the second output channel are the signal input and output channels respectively.

- Also, the other two input channels are noise input channels and the other two output channels are noise output channels.

- Note that unlike the case of a phase-insensitive quantum amplifier, it is straightforward to verify that it not possible to achieve a non-reciprocal phase-insensitive quantum amplifier with only a single noise channel on the input and output.

- The noise output channels are not used in the operation of the amplifier but are included for consistency with the physical realizability theory for quantum linear systems.
- As with any quantum linear system, each input and output channel consists of two quadratures.
- Hence, the transfer function matrix $\bar{G}(s)$ is a six-by-six transfer function matrix.
- In order to define a non-reciprocal phase-insensitive quantum amplifier, a physically realizable transfer function matrix $\bar{G}(s)$ should satisfy certain gain, non-reciprocal and phase-insensitivity properties over a specified frequency range.
- These properties will be formally defined below.

- We write the transfer function $\bar{G}(s)$ in “doubled-up” form, specifying both quadratures of each input and output channels as follows:

$$\left[\begin{array}{c} s^{out} \\ w_1^{out} \\ w_2^{out} \\ \hline s^{out*} \\ w_1^{out*} \\ w_2^{out*} \end{array}\right] = \bar{G} \left[\begin{array}{c} s^{in} \\ w_1^{in} \\ w_2^{in} \\ \hline s^{in*} \\ w_1^{in*} \\ w_2^{in*} \end{array}\right]$$

$$= \left[\begin{array}{cc} G & H \\ H^{\#} & G^{\#} \end{array}\right] \left[\begin{array}{c} s^{in} \\ w_1^{in} \\ w_2^{in} \\ \hline s^{in*} \\ w_1^{in*} \\ w_2^{in*} \end{array}\right].$$

- Furthermore, we write

$$G = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & G_{22} \end{bmatrix}, \quad H = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & H_{22} \end{bmatrix}.$$

- Here $g_{11},\ h_{11} \in \mathbb{C}$; $g_{12},\ h_{12} \in \mathbb{C}^{1\times 2}$; $g_{21},\ h_{21} \in \mathbb{C}^{2\times 1}$; $G_{22},\ H_{22} \in \mathbb{C}^{2\times 2}$.
- This involves grouping the two noise inputs together along with the corresponding outputs.
- In this description, the variable s^{in} represents the amplifier input signal and w_1^{out} represents the amplifier output signal.

Definition 5 *A physically realizable transfer function matrix* $\bar{G}(s)$ *of the form above is said to be* **phase-insensitive** *at frequency* ω *if*

$$[1 \;\; 0]\, h_{21}(j\omega) = 0.$$

Definition 6 *A physically realizable transfer function matrix* $\bar{G}(s)$ *of the form above is said to be* **non-reciprocal** *at frequency* ω *if*

$$g_{11}(j\omega) = 0, \;\; h_{11}(j\omega) = 0.$$

- In addition, our non-reciprocal phase-insensitive quantum amplifier will be required to have a specified complex gain g at frequency ω.
- This is represented by the gain constraint

$$[1 \;\; 0]\, g_{21}(j\omega) = g.$$

- We will be mostly concerned with the non-reciprocal and phase-insensitive properties at DC and hence, we will usually drop the frequency specification.
- Also, we will be concerned with the corresponding *noise* squared amplitude

$$N = \begin{bmatrix} 1 & 0 \end{bmatrix} G_{22}(j\omega)G_{22}(j\omega)^\dagger \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \end{bmatrix} H_{22}(j\omega)H_{22}(j\omega)^\dagger \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

at a given frequency ω (usually DC).

Main Results

Theorem 4 *At any frequency ω, the transfer function matrix defined by*

$$g_{11} = 0; g_{12} = \begin{bmatrix} 0 & 1 \end{bmatrix}; g_{21} = \begin{bmatrix} g \\ 0 \end{bmatrix}; G_{22} = \begin{bmatrix} 0 & 0 \\ g & 0 \end{bmatrix};$$

$$h_{11} = 0; h_{12} = \begin{bmatrix} 0 & 0 \end{bmatrix}; h_{21} = \begin{bmatrix} 0 \\ \sqrt{g^2 - 1} \end{bmatrix}; H_{22} = \begin{bmatrix} \sqrt{g^2 - 1} & 0 \\ 0 & 0 \end{bmatrix};$$

satisfies the physical realizability condition and the conditions required for a non-reciprocal, phase-insensitive quantum amplifier with real gain $g > 1$.
Moreover, the corresponding contribution of the noise inputs w_1^{in} and w_2^{in} to noise covariance of the output signal w_1^{out} is given by

$$N = g^2 - 1.$$

This is the minimal noise covariance for any non-reciprocal, phase-insensitive quantum amplifier with real gain g. Again this optimal amplifier is not unique.

- The proof of this theorem follows by straightforward substitution.
- To apply this theorem, we use the following version of the Shale decomposition for 6×6 complex matrices.

Lemma 3 *Consider a 6×6 complex matrix $\bar{G}$ of the form above satisfying the physical realizability condition. Then there exists a real diagonal matrix $R = \begin{bmatrix} r_1 & 0 & 0 \\ 0 & r_2 & 0 \\ 0 & 0 & r_3 \end{bmatrix}$ and 3×3 unitary matrices S_1 and S_2 such that*

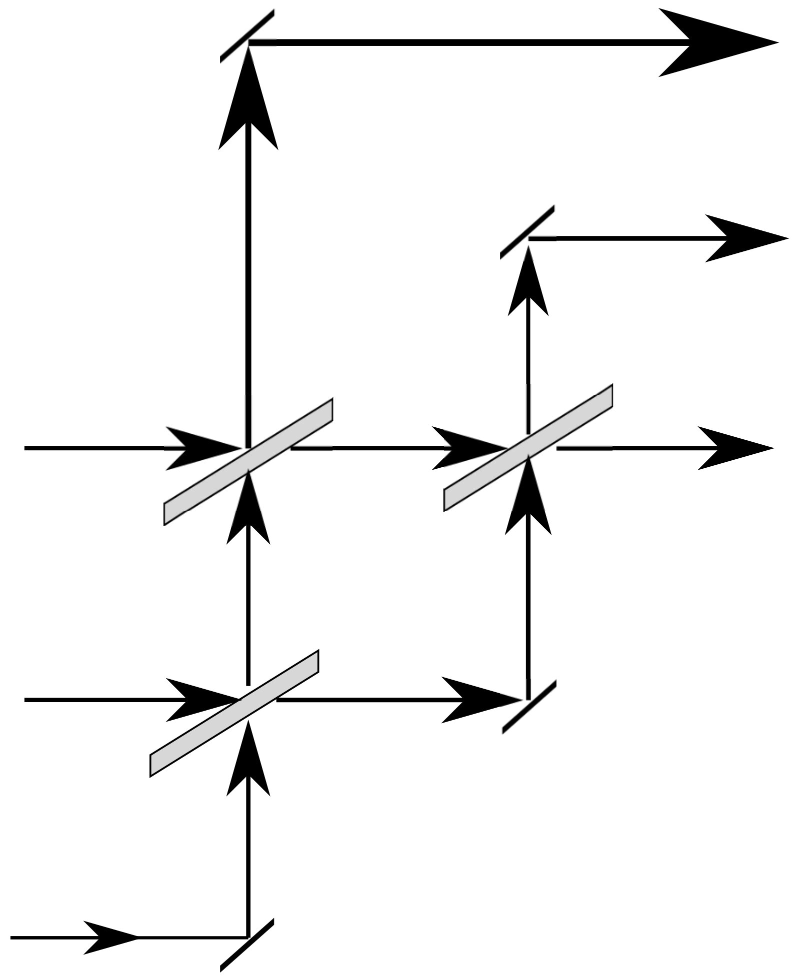
$$\bar{G} = \begin{bmatrix} S_1 & 0 \\ 0 & S_1^{\#} \end{bmatrix} \begin{bmatrix} -\cosh(R) & -\sinh(R) \\ -\sinh(R) & -\cosh(R) \end{bmatrix} \begin{bmatrix} S_2 & 0 \\ 0 & S_2^{\#} \end{bmatrix}.$$

- This lemma shows that the problem of physically realizing the three channel DC gain transfer function matrix $\bar{G}$ can be reduced to the problem of physically realizing each of the single channel transfer function matrices

$$\begin{aligned}\bar{G}_1 &= \begin{bmatrix} -\cosh(r_1) & -\sinh(r_1) \\ -\sinh(r_1) & -\cosh(r_1) \end{bmatrix}, \\ \bar{G}_2 &= \begin{bmatrix} -\cosh(r_2) & -\sinh(r_2) \\ -\sinh(r_2) & -\cosh(r_2) \end{bmatrix}, \\ \bar{G}_3 &= \begin{bmatrix} -\cosh(r_3) & -\sinh(r_3) \\ -\sinh(r_3) & -\cosh(r_3) \end{bmatrix}.\end{aligned}$$

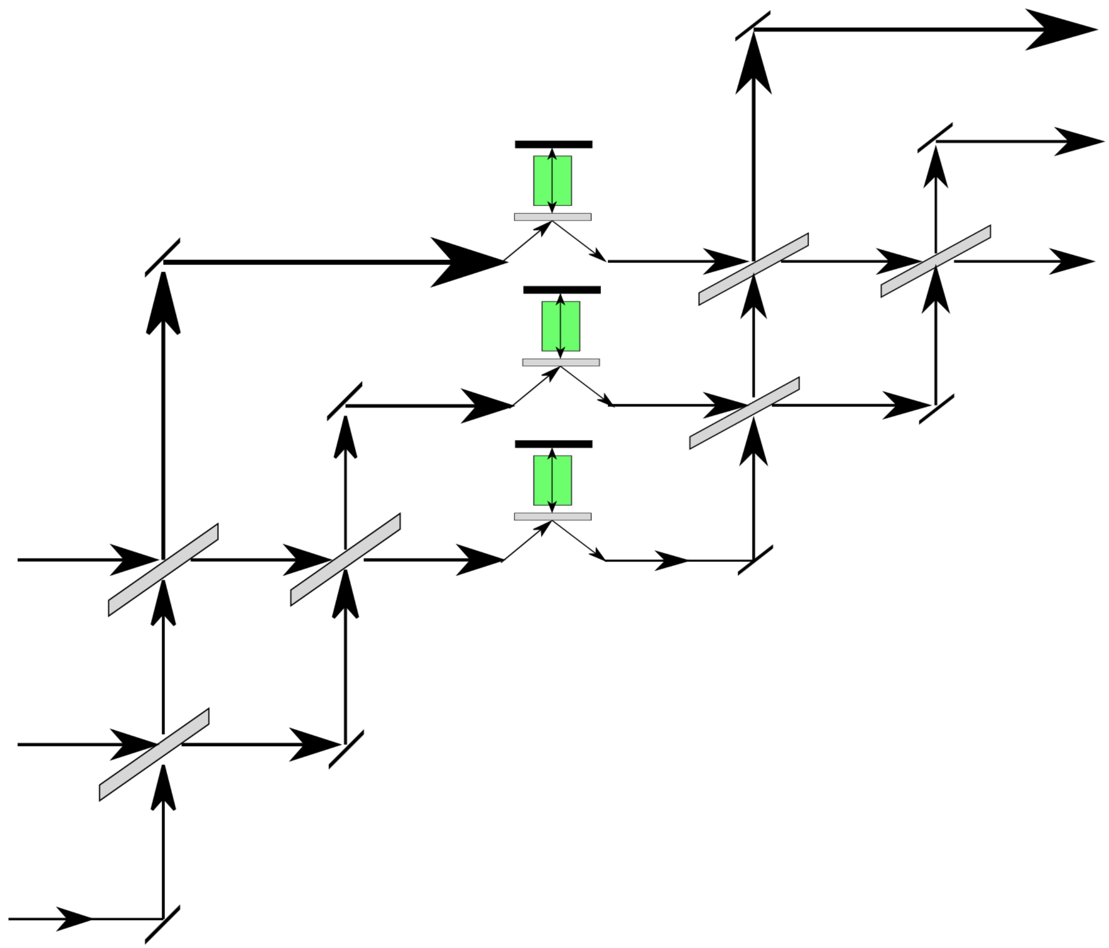
- Then, the unitary transfer matrices $\begin{bmatrix} S_1 & 0 \\ 0 & S_1^\# \end{bmatrix}$, and $\begin{bmatrix} S_2 & 0 \\ 0 & S_2^\# \end{bmatrix}$ can be physically implemented using beamsplitters.

- Indeed, it is straightforward show that any matrix of the form $\begin{bmatrix} S & 0 \\ 0 & S^\# \end{bmatrix}$ where S is a 3×3 unitary matrix can be implemented using a network of three beamsplitters as shown below.



Beamsplitter implementation of a 3×3 unitary matrix.

- Two such beamsplitter networks are combined with three squeezers to give the following complete optical implementation of a non-reciprocal phase-insensitive amplifier.



- Using the above theorems and lemmas, we obtain the following theorem.

Theorem 5

- *Given any desired quantum non-reciprocal phase-insensitive amplifier DC gain* $g > 1$*, there exists a corresponding physically realizable linear quantum system which achieves this DC gain and introduces the amount of DC quantum noise defined by* $N = g^2 - 1$.
- *This is the minimal amount of noise needed for a quantum phase-insensitive amplifier with DC gain* $g > 1$.
- *Furthermore, this transfer function matrix satisfies the DC phase-insensitivity condition and the non-reciprocal condition.*
- *In addition, the parameters in this linear quantum system can be chosen to achieve a specified bandwidth over which the above conditions will hold approximately.*
- *Finally, this system can be constructed from a collection of six beamsplitters and three stable dynamic squeezers.*

Conclusions

- We have used ideas from linear systems theory to design optimal quantum amplifiers.
- The notion of physical realizability enables us to connect ideas from linear systems theory to quantum problems.
- In the case of phase-insensitive amplifiers, quantum mechanics imposes fundamental limits on the amount of noise contributed by the amplifier and not the bandwidth.
- However, in practice, the bandwidth will be limited by technical factors relating to the type of physical implementation being used.
- We showed how optimal phase insensitive amplifiers could be implemented in both the optical case and in the superconducting microwave case.

- We also considered optimal non-reciprocal phase-insensitive amplifiers.
- It was found that the optimal noise level in this case is the same as in the case of optimal phase-insensitive amplifiers.
- However, the proposed optimal non-reciprocal phase-insensitive amplifier is more complicated than the proposed optimal phase-insensitive amplifier.
- Also, an optical implementation of an optimal non-reciprocal phase-insensitive amplifier was presented.
- Although similar amplifiers to those presented here are known in the physics literature, the systems theory approach provides a more systematic way of dealing with the problem of optimal quantum amplifier design.